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The critical exponent: a novel graph invariant

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Abstract. A surprising result of FitzGerald and Horn (1977) shows that $A^{\circ\alpha} := (a_{ij}^{\alpha})$ is positive semidefinite (p.s.d.) for every entrywise nonnegative $n \times n$ p.s.d. matrix $A = (a_{ij})$ if and only if α is a positive integer or $\alpha \geq n-2$. Given a graph G, we consider the refined problem of characterizing the set $\mathcal{H}_G$ of entrywise powers preserving positivity for matrices with a zero pattern encoded by G. Using algebraic and combinatorial methods, we study how the geometry of G influences the set $\mathcal{H}_G$. Our treatment provides new and exciting connections between combinatorics and analysis, and leads us to introduce and compute a new graph invariant called the *critical exponent*.

Résumé. Un résultat surprenant de FitzGerald et Horn (1977) démontre que $A^{\circ\alpha} := (a_{ij}^{\alpha})$ est semi-définie positive pour chaque matrice semi-définie positive de dimension n avec des entrées non-négatives si et seulement si α est un entier positif ou $\alpha \geq n-2$. Pour un graph G donné, nous considérons une généralization naturelle du problème en étudiant l'ensemble $\mathcal{H}_G$ de puissances préservant la positivité des matrices ayant une structure de zéros encodée par G. À l'aide de méthodes algébriques et combinatoires, nous analysons de quelle façon la géometrie du graph G détermine l'ensemble $\mathcal{H}_G$. Notre travail fournit de nouvelles connexions excitantes entre la combinatoire et l'analyse, et nous mène à définir et calculer un nouvel invariant que l'on nomme l'*exposant critique* d'un graph.

Keywords: Matrices with structure of zeros, chordal graphs, entrywise positive maps, positive semidefiniteness, Loewner ordering, fractional Schur powers

1 Introduction and main results

Let $\mathbb{N}$ denote the set of positive integers. Given $n \in \mathbb{N}$ and $I \subset \mathbb{R}$, let $\mathbb{P}_n(I)$ denote the set of symmetric positive semidefinite $n \times n$ matrices with entries in I. Given two

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$n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, their Hadamard (or Schur, or entrywise) product, denoted by $A \circ B$, is defined by $A \circ B := (a_{ij} b_{ij})$. Note that $A \circ B$ is a principal submatrix of the tensor product $A \otimes B$. As a consequence, if A and B are positive semidefinite, then so is $A \circ B$. This result is known in the literature as the *Schur product theorem*.

Given $\alpha \in \mathbb{R}$, we denote the entrywise αth power of a matrix A with nonnegative entries by $A^{\circ\alpha} := (a_{ij}^{\alpha})$, where we define $0^{\alpha} := 0$ for all α. By the Schur product theorem, $A^{\circ k}$ is positive (semi)definite for all positive (semi)definite matrices A and all $k \in \mathbb{N}$. It is natural to ask if other real powers have the same property. This problem was resolved by FitzGerald and Horn, who revealed a surprising phase transition phenomenon.

Theorem 1.1 (FitzGerald–Horn [11])**.** *Let $n \geq 2$.*

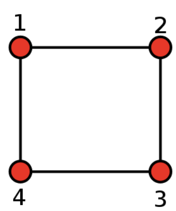
1) If $\alpha \in \mathbb{N}$ or $\alpha \geq n-2$, then $A^{\circ\alpha} \in \mathbb{P}_n(\mathbb{R})$ for all $A \in \mathbb{P}_n([0,\infty))$.

2) If $\alpha \in (0, n-2) \setminus \mathbb{N}$, then there exists a matrix $A \in \mathbb{P}_n([0,\infty))$ such that $A^{\circ\alpha} \notin \mathbb{P}_n(\mathbb{R})$.

We now refine the above problem by restricting the set of matrices to those with a given sparsity structure. Given a finite undirected simple graph $G = (V, E)$ with nodes $V = \{1, 2, \dots, n\}$, and a subset $I \subset \mathbb{R}$, define

$$\mathbb{P}_G(I) := \{A \in \mathbb{P}_n(I) : a_{ij} = 0 \ \forall (i,j) \notin E, \ i \neq j\}. \tag{1.1}$$

We denote $\mathbb{P}_G(\mathbb{R})$ by $\mathbb{P}_G$. All graphs below are assumed to be finite and simple.



$$\begin{pmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & * & * & * \\ * & 0 & * & * \end{pmatrix}$$

Figure 1.1: $\mathbb{P}_G$ for G a 4-cycle. Entries with an asterisk are not constrained.

The goal of this paper is to study the set of entrywise powers preserving positivity on the set $\mathbb{P}_G$. Such structured matrices arise naturally in various subfields of mathematics, including combinatorial matrix analysis [1, 8], spectral graph theory [10], and graphical models [21]. As we explain below, such matrices and their entrywise transforms are also of importance in modern-day applications in high-dimensional covariance estimation, making the problem at once classically motivated as well as timely.

We now establish further notation. Note that when $\alpha \notin \mathbb{N}$, and A is a real matrix, $A^{\circ\alpha}$ is not always well-defined. Thus, we follow Hiai [20] and work with the odd and even extensions to $\mathbb{R}$ of the power functions. Define

$$\psi_\alpha(x) := \operatorname{sgn}(x)|x|^\alpha, \qquad \phi_\alpha(x) := |x|^\alpha, \qquad \forall\, x \in \mathbb{R} \setminus \{0\}, \tag{1.2}$$

and $\psi_\alpha(0) = \phi_\alpha(0) := 0$. Given $f : \mathbb{R} \to \mathbb{R}$, and $A = (a_{ij})$, define $f[A] := (f(a_{ij}))$. We now introduce the main objects of study in this paper.

Definition 1.2. *Let $n \geq 2$ and let $G = (V, E)$ be a simple graph on $V = \{1, \dots, n\}$. We define:*

$$\mathcal{H}_G := \{\alpha \in \mathbb{R} : A^{\circ\alpha} \in \mathbb{P}_G \text{ for all } A \in \mathbb{P}_G([0,\infty))\},$$
$$\mathcal{H}_G^{\psi} := \{\alpha \in \mathbb{R} : \psi_\alpha[A] \in \mathbb{P}_G \text{ for all } A \in \mathbb{P}_G(\mathbb{R})\},$$
$$\mathcal{H}_G^{\phi} := \{\alpha \in \mathbb{R} : \phi_\alpha[A] \in \mathbb{P}_G \text{ for all } A \in \mathbb{P}_G(\mathbb{R})\}.$$

Theorem 1.1 thus shows: $\mathcal{H}_{K_n} = \mathbb{N} \cup [n-2, \infty)$. The sets $\mathcal{H}_{K_n}^{\psi}$ and $\mathcal{H}_{K_n}^{\phi}$ have also been computed, and exhibit similar phase transitions.

Theorem 1.3 (FitzGerald–Horn [11], Bhatia–Elsner [6], Hiai [20], Guillot–Khare–Rajaratnam [14]). *Let $n \geq 2$. The $\mathcal{H}$-sets of powers preserving positivity for $G = K_n$ are:*

$$\mathcal{H}_{K_n} = \mathbb{N} \cup [n-2, \infty), \qquad \mathcal{H}_{K_n}^{\psi} = (-1 + 2\mathbb{N}) \cup [n-2, \infty), \qquad \mathcal{H}_{K_n}^{\phi} = 2\mathbb{N} \cup [n-2, \infty).$$

(See [14] for more details.) Theorem 1.3 demonstrates that there is a threshold value above which every power function x^α, ψ_α, or ϕ_α preserves positivity on $\mathbb{P}_n([0,\infty))$ or $\mathbb{P}_n(\mathbb{R})$, when applied entrywise. The threshold is commonly referred to as the *critical exponent* for preserving positivity. We now extend this notion to all graphs.

Definition 1.4. *Given a graph G, define the* (Hadamard) critical exponents of G *to be*

$$CE_H(G) := \min\{\alpha \in \mathbb{R} : A \in \mathbb{P}_G([0,\infty)) \Rightarrow A^{\circ\beta} \in \mathbb{P}_G \text{ for every } \beta \geq \alpha\},$$
$$CE_H^{\psi}(G) := \min\{\alpha \in \mathbb{R} : A \in \mathbb{P}_G(\mathbb{R}) \Rightarrow \psi_\alpha[A] \in \mathbb{P}_G \text{ for every } \beta \geq \alpha\},$$
$$CE_H^{\phi}(G) := \min\{\alpha \in \mathbb{R} : A \in \mathbb{P}_G(\mathbb{R}) \Rightarrow \phi_\alpha[A] \in \mathbb{P}_G \text{ for every } \beta \geq \alpha\}.$$

These critical exponents appear to be new graph invariants, not previously studied in the literature. Note that since every graph $G = (V, E)$ is contained in a complete graph, the critical exponents of G are well defined by Theorem 1.3, and bounded above by $|V| - 2$. However, computing critical exponents is a challenging problem at the intersection of graph theory, analysis, and matrix theory. Indeed, the critical exponents are not known for many families of non-complete graphs, and provide interesting avenues of research in combinatorial matrix analysis, which also have the potential to impact other areas.

We now state our main result. To do so, we first recall the notion of *chordal graphs*. These are precisely the graphs G in which every cycle of length 4 or more has a chord. Chordal graphs are prominent in mathematics as well as applications. They are also known as decomposable graphs, triangulated graphs, and rigid circuit graphs; have a rich structure, and include several well-known examples of graphs (see Table 2.1). Chordal graphs play a fundamental role in multiple areas including the matrix completion problem [5, 13, 23], maximum likelihood estimation in the theory of Markov random fields [21, Section 5.3], and perfect Gaussian elimination [12].

Let $K_n^{(1)}$ be the complete graph on n vertices with one edge missing. Then we have:

Theorem 1.5 (Main result)**.** *Let G be any chordal graph with at least 2 vertices and let r be the largest integer such that $K_r^{(1)}$ is a subgraph of G. Then*

$$\mathcal{H}_G = \mathbb{N} \cup [r-2, \infty), \quad \mathcal{H}_G^{\psi} = (-1 + 2\mathbb{N}) \cup [r-2, \infty), \quad \mathcal{H}_G^{\phi} = 2\mathbb{N} \cup [r-2, \infty). \quad (1.3)$$

In particular, $CE_H(G) = CE_H^{\psi}(G) = CE_H^{\phi}(G) = r - 2$.

Our main result extends the previous work in Theorem 1.3 to the important family of chordal graphs. Moreover, it shows how the problem of finding powers preserving positivity is solvable using combinatorial techniques.

We conclude this section with some remarks. First, the cones $\mathbb{P}_G$ of structured matrices naturally arise in applications, as (inverse) covariance/correlation matrices with an underlying graphical model [21]. Powering such matrices entrywise is a way to regularize them in high-dimensional probability and statistics. This procedure often improves their properties (e.g. condition number) and helps separate signal from noise – see [7, 22, 24] for more details. Theorem 1.5 and related results are relevant in this context. For instance, we show in recent works [15, 16] that the critical exponent $CE_H(G)$ of any tree or bipartite graph G is 1. As a consequence, unlike in the unconstrained case of $\mathbb{P}_n$ (i.e., K_n), families of sparse as well as dense graphs $G = (V, E)$ can have very small critical exponents that do not grow with V. This is important as such small powers can regularize matrices, yet minimally modify their entries.

This work is an extended abstract of [15]. Understanding which powers preserve positivity is part of a broad program by the authors; see [4, 2, 3, 14, 15, 16, 17, 18, 19] and the references therein. Our work has yielded surprising connections to other areas such as Schur polynomials and symmetric function theory; see [4, 2] for more details.

2 Proof of the main result

In this section we provide the main ideas used to prove Theorem 1.5. Complete proofs as well as other ramifications of the results in this paper can be found in [15].

We begin by recalling some properties of chordal graphs (see e.g. [9, Chapter 5.5], [12, Chapter 4]). Given a graph $G = (V, E)$, and $C \subset V$, denote by G_C the subgraph of G induced by C. A *clique* in G is a complete induced subgraph of G. A subset $C \subset V$ *separates* $A \subset V$ from $B \subset V$ if every path from a vertex in A to a vertex in B intersects C. A partition (A, C, B) of subsets of V is a *decomposition* of G if G_C is a clique and C separates A from B (see Figure 2.1). A graph G is *decomposable* if either G is complete, or if there exists a decomposition (A, C, B) of G such that $G_{A \cup C}$ and $G_{B \cup C}$ are decomposable.

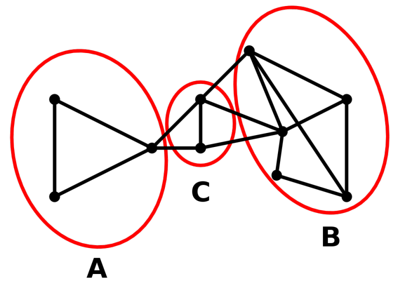


Figure 2.1: Decomposition of a graph.

Let G be a graph and let $B_1, \ldots, B_k$ be a sequence of subsets of vertices of G. Define:

$$H_j := B_1 \cup \cdots \cup B_j, \qquad R_j = B_j \setminus H_{j-1}, \qquad S_j = H_{j-1} \cap B_j, \qquad 1 \leq j \leq k, \tag{2.1}$$

and $H_0 := \emptyset$. The sets H_j, R_j, and S_j are respectively called the *histories, residuals, and separators* of the sequence. The sequence $B_1, \ldots, B_k$ is said to be a *perfect ordering* if:

1. For all $1 < i \leq k$, there exists $1 \leq j < i$ such that $S_i \subset B_j$; and
2. The sets S_i induce complete graphs for all $1 \leq i \leq k$.

Decompositions and perfect orderings provide important characterizations of chordal graphs, as summarized in Theorem 2.1.

Theorem 2.1 ([21, Chapter 2])**.** *Given a graph $G = (V, E)$, the following are equivalent:*

1. *G is chordal (i.e., each cycle with 4 vertices or more in G has a chord).*
2. *G is decomposable.*
3. *The maximal cliques of G admit a perfect ordering.*

We now provide a correspondence between decompositions of a graph G, and decompositions of matrices in the associated cone $\mathbb{P}_G$. In the statement of the result and below, given a graph G and an induced subgraph G', we identify $\mathbb{P}_{G'}(I)$ with a subset of $\mathbb{P}_G(I)$ when convenient, via the assignment $M \mapsto M \oplus \mathbf{0}_{(V(G)\setminus V(G')) \times (V(G)\setminus V(G'))}$.

Lemma 2.2. *Let $G = (V, E)$ be a graph with a decomposition (A, C, B) of V, and let M be a symmetric matrix. Assume the principal submatrices M_{AA} and M_{BB} of M are invertible. Then the following are equivalent:*

1. *$M \in \mathbb{P}_G$.*
2. *$M = M_1 + M_2$ for some matrices $M_1 \in \mathbb{P}_{G_{A\cup C}}$ and $M_2 \in \mathbb{P}_{G_{B\cup C}}$.*

The proof of Lemma 2.2 requires working with Schur complements. Recall that a symmetric block matrix $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ with C invertible is positive definite if and only if C is positive definite, and the Schur complement of C in M

$$M/C := A - BC^{-1}B^T$$

is positive definite. Similarly, if A is invertible, then M is positive definite if and only if A is positive definite and the Schur complement $M/A := C - B^T A^{-1} B$ is positive definite.

Proof of Lemma 2.2. Clearly $(2) \implies (1)$. Conversely, write $M \in \mathbb{P}_G$ in block form as

$$M = \begin{pmatrix} M_{AA} & M_{AC} & 0 \\ M_{AC}^T & M_{CC} & M_{CB} \\ 0 & M_{CB}^T & M_{BB} \end{pmatrix}.$$

Then $M = M_1 + M_2$, with

$$M_1 := \begin{pmatrix} M_{AA} & M_{AC} & 0 \\ M_{AC}^T & M_{AC}^T M_{AA}^{-1} M_{AC} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & M_{CC} - M_{AC}^T M_{AA}^{-1} M_{AC} & M_{CB} \\ 0 & M_{CB}^T & M_{BB} \end{pmatrix}.$$

Using properties of Schur complements, we easily verify that $M_1 \in \mathbb{P}_{G_{A\cup C}}, M_2 \in \mathbb{P}_{G_{B\cup C}}$. □

Note that Lemma 2.2 also provides information about the extreme points of the convex cone $\mathbb{P}_G$ when G is decomposable. As we show below, the problem of understanding the geometry of $\mathbb{P}_G$ is closely related to the computation of the $\mathcal{H}$ sets in Definition 1.2.

Remark 2.3. *When G has a decomposition (A, C, B) and $M \in \mathbb{P}_G$, then M also factors as*

$$M = \begin{pmatrix} M_{AA} & 0 & 0 \\ M_{AC}^T & \mathrm{Id}_{|C|} & M_{CB} \\ 0 & 0 & M_{BB} \end{pmatrix} \begin{pmatrix} M_{AA}^{-1} & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & M_{BB}^{-1} \end{pmatrix} \begin{pmatrix} M_{AA} & 0 & 0 \\ M_{AC}^T & \mathrm{Id}_{|C|} & M_{CB} \\ 0 & 0 & M_{BB} \end{pmatrix}^T, \tag{2.2}$$

where Id_k denotes the $k \times k$ identity matrix, and $S := M_{CC} - M_{AC}^T M_{AA}^{-1} M_{AC} - M_{CB} M_{BB}^{-1} M_{CB}^T$. We will make use of this factorization later.

Lemma 2.2 provides a powerful technique to verify when functions preserve positivity on $\mathbb{P}_G$ when applied entrywise. Indeed, suppose G is a graph with a decomposition (A, C, B). Let $M \in \mathbb{P}_G$. Write $M = M_1 + M_2$ as in Lemma 2.2, with $M_1 \in \mathbb{P}_{G_{A\cup C}}$ and $M_2 \in \mathbb{P}_{G_{B\cup C}}$. Given a real function $f : \mathbb{R} \to \mathbb{R}$, recall that we denote by $f[M]$ the matrix $(f(m_{ij}))$. Now if $f[M] - f[M_1] - f[M_2]$ is positive semidefinite, then $f[M] \in \mathbb{P}_G$ if f preserves positivity on $\mathbb{P}_{G_{A\cup C}}$ and $\mathbb{P}_{G_{B\cup C}}$. Thus, we introduce the following notion.

Definition 2.4. *Given a graph G and a function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$, we say that $f[-]$ is* Loewner super-additive on $\mathbb{P}_G(\mathbb{R})$ *if $f[A + B] - f[A] - f[B] \in \mathbb{P}_G(\mathbb{R})$ for $A, B \in \mathbb{P}_G(\mathbb{R})$.*

Note that this notion coincides with the usual notion of super-additivity on $[0, \infty)$ when G has only one vertex.

The above discussion shows that a function preserves positivity on $\mathbb{P}_G$ if it preserves positivity on $\mathbb{P}_{G_{A\cup C}}$ and $\mathbb{P}_{G_{B\cup C}}$ and is Loewner super-additive on $\mathbb{P}_{G_C}$. Theorem 2.5 below shows that the converse also holds under certain assumptions.

Theorem 2.5. *Let $G = (V, E)$ be a graph with a decomposition (A, C, B), and let $f : \mathbb{R} \to \mathbb{R}$.*

1. *If $f[-]$ preserves positivity on $\mathbb{P}_{G_{A\cup C}}$ and on $\mathbb{P}_{G_{B\cup C}}$ and is Loewner super-additive on $\mathbb{P}_{G_C}$ then $f[-]$ preserves positivity on $\mathbb{P}_G$.*
2. *Conversely, if $f = \psi_\alpha$ or $f = \phi_\alpha$ and $f[-]$ preserves positivity on $\mathbb{P}_G$, then $f[-]$ is Loewner super-additive on $\mathbb{P}_{G_{C'}}$ for every clique $C' \subset C$ for which there exist vertices $v_1 \in A, v_2 \in B$ that are adjacent to every $v \in C'$ (see Figure 2.2).*

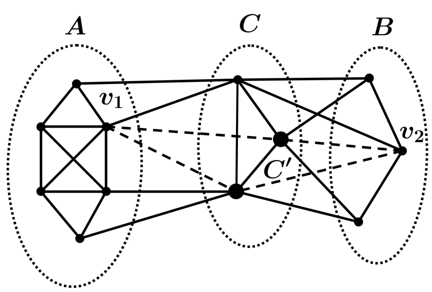


Figure 2.2: Illustration of the second part of Theorem 2.5.

To prove Theorem 2.5, we recall previous work on Loewner super-additive functions.

Theorem 2.6 (Guillot, Khare, and Rajaratnam [14, Theorem 5.1])**.** *Given an integer $n \geq 2$, the sets of entrywise powers $\alpha \in \mathbb{R}$, such that the functions $f_\alpha(x) = x^\alpha, \psi_\alpha(x), \phi_\alpha(x)$ are Loewner super-additive maps on $\mathbb{P}_n(\mathbb{R})$ are, respectively,*

$$\mathbb{N} \cup [n, \infty), \qquad (-1 + 2\mathbb{N}) \cup [n, \infty), \qquad 2\mathbb{N} \cup [n, \infty).$$

Moreover, the same results hold if $\mathbb{P}_n(\mathbb{R})$ is replaced by the set of rank one matrices in $\mathbb{P}_n(\mathbb{R})$.

Sketch of the proof of Theorem 2.5. We only prove the second part (see [15] for a complete proof). Suppose $f = \psi_\alpha$ or ϕ_α for $\alpha \in \mathbb{R}$, and $f[-]$ preserves positivity on $\mathbb{P}_G$. Then clearly $f[-]$ preserves positivity on $\mathbb{P}_{G_{A\cup S}}$ and $\mathbb{P}_{G_{B\cup S}}$. Moreover, suppose there exist $v_1 \in A$, $v_2 \in B$, and a clique $C' \subset C$ of size m such that v_1 and v_2 are adjacent to every vertex in C'. Assume, without loss of generality, that the vertices of G are labelled in the following order: v_1, the m vertices in C', v_2, and the remaining vertices of G. Now given vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ and a $m \times m$ symmetric matrix M, define the matrix

$$W(\mathbf{u}, \mathbf{v}, M) := \begin{pmatrix} 1 & \mathbf{u}^T & 0 \\ \mathbf{u} & M & \mathbf{v} \\ 0 & \mathbf{v}^T & 1 \end{pmatrix}. \tag{2.3}$$

Then $W(\mathbf{u}, \mathbf{v}, \mathbf{u}\mathbf{u}^T + \mathbf{v}\mathbf{v}^T) \oplus \mathbf{0}_{|V|-(m+2)} \in \mathbb{P}_G(\mathbb{R})$, so by the assumptions on f, we conclude that $f[W(\mathbf{u}, \mathbf{v}, \mathbf{u}\mathbf{u}^T + \mathbf{v}\mathbf{v}^T)] = W(f[\mathbf{u}], f[\mathbf{v}], f[\mathbf{u}\mathbf{u}^T + \mathbf{v}\mathbf{v}^T]) \in \mathbb{P}_{m+2}(\mathbb{R})$. Using the factorization (2.2), we conclude that

$$f[\mathbf{u}\mathbf{u}^T + \mathbf{v}\mathbf{v}^T] - f[\mathbf{u}]f[\mathbf{u}^T] - f[\mathbf{v}]f[\mathbf{v}^T] = f[\mathbf{u}\mathbf{u}^T + \mathbf{v}\mathbf{v}^T] - f[\mathbf{u}\mathbf{u}^\mathbf{T}] - f[\mathbf{v}\mathbf{v}^T] \geq 0. \tag{2.4}$$

Thus $f = \psi_\alpha, \phi_\alpha$ is Loewner super-additive on rank one matrices in $\mathbb{P}_m$. By Theorem 2.6, we conclude that f is Loewner super-additive on all of $\mathbb{P}_m$. □

We now provide a sketch of the proof of Theorem 1.5 (see [15] for full details).

Sketch of the proof of Theorem 1.5. Suppose G is a chordal graph, which we may assume to be connected. Let r be as in the statement of the theorem. One can show that Theorem 1.3 still holds when K_n is replaced by $K_n^{(1)}$. As a result,

$$\mathcal{H}_G \subset \mathbb{N} \cup [r-2, \infty), \quad \mathcal{H}_G^\psi \subset (-1 + 2\mathbb{N}) \cup [r-2, \infty), \quad \mathcal{H}_G^\phi = 2\mathbb{N} \cup [r-2, \infty). \tag{2.5}$$

We now prove the reverse inclusions. By Theorem 2.1, the maximal cliques of G admit a perfect ordering $\{C_1, \dots, C_k\}$. We will prove the reverse inclusions in (2.5) by induction on k. If $k = 1$, then G is complete and the inclusions clearly hold by Theorem 1.3. Suppose the result holds for all chordal graphs with $k = l$ maximal cliques, and let G be a graph with $k = l + 1$ maximal cliques. For $1 \leq j \leq k$, define

$$H_j := C_1 \cup \cdots \cup C_j, \qquad C_j = C_j \setminus H_{j-1}, \qquad S_j = H_{j-1} \cap C_j \tag{2.6}$$

as in (2.1). By [21, Lemma 2.11], the triplet (H_{k-1}, S_k, R_k) is a decomposition of G. Let $\alpha \in [r-2, \infty)$. By the induction hypothesis, the three αth power functions preserve positivity on $\mathbb{P}_{G_{H_{k-1} \cup S_k}} = \mathbb{P}_{G_{H_{k-1}}}$. Moreover, since $\alpha \geq r-2$, they also preserve positivity on $\mathbb{P}_{G_{C_k \cup S_k}} = \mathbb{P}_{G_{C_k}}$. We now claim that $r \geq |S_k| + 2$. Clearly, $|S_k| \leq r$ since S_k is complete. If $|S_k| = r$, then C_k is contained in one of the previous cliques, which is a contradiction. Suppose instead that $|S_k| = r - 1$. Since $\{C_1, \dots, C_k\}$ is a perfect ordering, $S_k \subset C_i$ for some $i < k$. Let $v \in C_i \setminus S_k$ and let $w \in R_k$. As v, w are adjacent to every $s \in S_k$, the subgraph of G induced by $S_k \cup \{v, w\}$ is isomorphic to $K_{r+1}^{(1)}$, which contradicts the definition of r. It follows that $r \geq |S_k| + 2$, as claimed. Now by Theorem 2.6, the αth power functions are Loewner super-additive on $\mathbb{P}_{S_k}$. Applying Theorem 2.5, we conclude that $\alpha \in \mathcal{H}_G^\psi, \mathcal{H}_G^\phi$, and hence $\alpha \in \mathcal{H}_G$. This concludes the proof of the theorem. □

The following corollary shows how to systematically compute the critical exponent of a chordal graph.

Corollary 2.7. *Suppose $G = (V, E)$ is chordal, $V = \{v_1, \dots, v_m\}$, and denote the maximal cliques in G by $C_1, \dots, C_n$. Define the "maximal clique matrix" of G to be $M(G) := (\mathbf{1}(v_i \in C_j))$. Then the critical exponent of G equals the largest entry of $M(G)^T M(G) - 2\,\mathrm{Id}_{|V|}$, i.e.,*

$$CE_H(G) = CE_H^{\psi}(G) = CE_H^{\phi}(G) = \max_{i,j}(u_i^T u_j - 2\delta_{i,j}), \tag{2.7}$$

where $u_1, \dots, u_n \in \{0,1\}^m$ are the columns of $M(G)$.

In particular, Corollary 2.7 can be used to compute the critical exponent of interval graphs, which are a well-known class of chordal graphs. Note that Theorem 2.5 can be used to compute the critical exponents of several other important graphs; see Table 2.1.

Graph G	$CE_H(G)$, $CE_H^{\psi}(G)$, $CE_H^{\phi}(G)$
Tree	1
Complete graph K_n	$n-2$
Minimal planar triangulation of C_n for $n \geq 4$	2
Apollonian graph, $n \geq 3$	$\min(3, n-2)$
Maximal outerplanar graph, $n \geq 3$	$\min(2, n-2)$
Band graph with bandwidth $d \leq n$	$\min(d, n-2)$
Split graph with maximal clique C	$\max(\lvert C\rvert - 2, \max\deg(V \setminus C))$

Table 2.1: Critical exponents of important families of chordal graphs with n vertices.

3 Non-chordal graphs: results and open problems

Computing the set of powers preserving positivity on $\mathbb{P}_G$ for general non-chordal graphs G still remains open. In this section, we mention recent results along this direction, and conclude by outlining several open questions.

As shown in Section 2, decompositions of graphs can be used to make reductions when computing critical exponents of chordal graphs. For non-chordal graphs, the decomposition process can still be iterated until components cannot be decomposed anymore. The resulting components are called the *prime components* of the graphs. The following result is akin to Theorem 2.5 for non-chordal graphs.

Theorem 3.1 ([15, Theorem 4.1])**.** *Let G be a graph with a perfect ordering $\{B_1, \dots, B_k\}$ of its prime components, and let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(0) = 0$. Define $s := \max_{i=1,\dots,k} |S_i|$, where S_i is defined as in (2.1). If $f[-]$ preserves positivity on $\mathbb{P}_{B_i}$ for all $1 \leq i \leq k$ and is Loewner super-additive on $\mathbb{P}_{K_s}$, then $f[-]$ preserves positivity on $\mathbb{P}_G$.*

3.1 Cycles and bipartite graphs

Chordal graphs are graphs without induced cycles of length 4 or more. A next natural step is thus to examine the case of cycles. In recent work [15], we show:

Theorem 3.2 ([15, Proposition 4.3])**.** *For all $n \geq 3$, $\mathcal{H}_{C_n} = \mathcal{H}^{\psi}_{C_n} = [1, \infty)$, and $\mathcal{H}^{\phi}_{C_4} = [2, \infty)$. Moreover, for $n > 4$, $[2, \infty) \subset \mathcal{H}^{\phi}_{C_n} \subset [1, \infty)$, with $1 \notin \mathcal{H}^{\phi}_{C_n}$ for n even.*

Note that Theorem 3.2 is in line with Theorem 1.5 as $r = 3$ is the biggest integer such that $K_r^{(1)}$ is contained in C_n.

Another very common family of non-chordal graphs is the bipartite graphs.

Theorem 3.3. *Suppose G is a connected bipartite graph with at least 3 vertices. Then,*

$$\mathcal{H}_G = [1, \infty), \qquad [2, \infty) \subset \mathcal{H}^{\phi}_G \subset [1, \infty), \qquad \{1\} \cup [3, \infty) \subset \mathcal{H}^{\psi}_G \subset [1, \infty).$$

If moreover $K_{2,2} \subset G \subset K_{2,m}$ for some $m \geq 2$, then

$$\mathcal{H}^{\phi}_G = [2, \infty), \qquad \{1\} \cup [2, \infty) \subset \mathcal{H}^{\psi}_G \subset [1, \infty).$$

Akin to Theorem 3.2, note that Theorem 3.3 is also in agreement with Theorem 1.5. Moreover, Theorem 3.3 has a surprising conclusion: broad families of dense graphs such as complete bipartite graphs can have small critical exponents that do not grow with the number of vertices. This has important applications in high-dimensional statistics: for appropriate structures of zeros, small powers can be used to minimally modify the entries of covariance matrices to improve their properties (see Introduction), while maintaining positivity. Note also that the result is in sharp contrast to the general case (Theorem 1.1), where there is no underlying structure of zeros.

3.2 Concluding remarks and open problems

The critical exponent of several other graphs (including coalescences of graphs and graphs obtained by pasting cycles to other graphs) were computed in [15]. However, the critical exponent is unknown for general graphs; it appears that new ideas in algebra, combinatorics, and convex geometry will be required to solve the question completely. We conclude this short paper by formulating some open problems that we hope will stimulate research at the intersection of these three areas.

Open problems:

1) Every graph G for which $CE_H(G)$ is currently known satisfies $CE_H(G) = r - 2$ where r is the largest integer such that G contains $K_r^{(1)}$ as a subgraph. Does this equality in fact hold for every graph?

2) It appears that the critical exponent of a graph is always an integer. Can this be proved directly without explicitly computing critical exponents?

3) If G' is obtained from G by adding a new vertex to G, and connecting it to every vertex of G, is it true that $CE_H(G') \leq CE_H(G) + 1$?

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